Poincaré duality & Thom class (after Hutchings notes §1, 4, 5)

Another interpretation of Euler class: assume M is a closed oriented mfld

Then we have Poincaré duality $H^i(M,\mathbb{Z}) \xrightarrow{\sim} H_{n-i}(M,\mathbb{Z})$

$$\alpha \longmapsto [M]\cap\alpha$$

Under Poincaré duality: $\begin{cases} \bullet \text{ cup product} \leftrightarrow \text{intersection of cycles.} \\ \bullet \text{ Euler class of } E\to M \leftrightarrow \text{zero set of a section of } E \end{cases}$

To make this rigorous, need to assume smoothness & transversality ...

- M oriented smooth mfld, dim. n; A, B oriented smooth submfds, dims. $n-i$ & $n-j$.

  Assume A, B intersect transversely ie. $\forall p\in A\cap B$, $T_pA + T_pB = T_pM$

  (this is a generic property, achieved by small isotopies)

  Then $A\cap B$ submfld of dim. $n-(i+j)$, $T_p(A\cap B) = T_pA\cap T_pB$.

  carries a natural orientation: induced by those of T_pM, T_pA, T_pB:

  choose oriented basis $u_1\dots u_{n-i-j}, v_1,\dots,v_j, w_1\dots w_i$ of T_pM st.

  $\quad$ —— " ——, $v_1\dots v_j$ oriented basis of T_pA

  $\quad$ —— " ——, $w_1\dots w_j$ oriented basis of T_pB

  then declare $u_1\dots u_{n-i-j}$ to be an oriented basis of $T_p(A\cap B)$

  Let $[A]^*\in H^i(M,\mathbb{Z})$ Poincaré dual to $[A]\in H_{n-i}(M,\mathbb{Z})$ (ie. $[M]\cap[A]^* = [A]$)

  $[B]^*\in H^j$ $\qquad$ $[B]\in H_{n-j}$

  $[A\cap B]^*\in H^{i+j}$ $\qquad$ $[A\cap B]\in H_{n-i-j}$

  <u>Thm:</u> $[A]^*\cup[B]^* = [A\cap B]^* \in H^{i+j}(M,\mathbb{Z})$.

  <u>Ex.:</u> for $\mathbb{CP}^n$, $H^2(\mathbb{CP}^n,\mathbb{Z})\ni\alpha = [\mathbb{CP}^{n-1}]^*$

  α^k = P. dual to intersection of k hyperplanes $\Rightarrow$ $\alpha^k = [\mathbb{CP}^{n-k}]^*$

  $\rightsquigarrow$ <u>intersection pairing</u> $\bullet$: $H_{n-i}\times H_{n-j}\to H_{n-i-j}$ $\qquad (a\cdot b)^* = a^*\cup b^*$.

  (eg if $i+j=n$, signed # of intersection pts).

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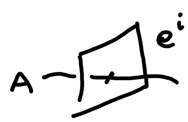
* <u>NB:</u> one way to think of P. duality: if $A\subset M$ transverse to a smooth cell decomp$^{\underline{n}}$,

$[A]^*$ = class rep$^{\underline{d}}$ by cocycle A^*: e^i i-cell $\longmapsto$ signed intersection # of A with e^i.



(then $A^*\cap[M]$:  gives chain $\sim A$).

This is more or less same as Thom class of normal bundle of A in M, recalling tubular nbd thm and embedding disc nbd $(D, D-A) \subset (M, M-A)$.
or equivalently $(D, \partial D) \subset (M, M-D)$.

* Let's first make the relation between <u>Thom class & P. duality</u> precise.

Namely: the Thom class is Poincaré dual to the <u>zero section</u>

B smooth closed oriented k-mfld, $E \to B$ smooth oriented rank n vector bundle
$D \subset E$ the disc bundle (= vectors of length ≤ 1 for some metric on E)

$\Rightarrow$ can view the Thom class $u \in H^n(E, E_0; \mathbb{Z}) \simeq H^n(E, E-D; \mathbb{Z}) \simeq H^n(D, \partial D; \mathbb{Z})$ by excision
(fiberwise: $H^n(\mathbb{R}^n, \mathbb{R}^n - 0, \mathbb{Z}) \simeq H^n(D^n, S^{n-1}; \mathbb{Z})$)

D is oriented (using: orientation of fiber $\oplus$ orientation of base, in that order)

Recall Poincaré duality gives an isom. $H^i(D, \partial D; \mathbb{Z}) \xrightarrow{\sim} H_{n+k-i}(D, \mathbb{Z})$
$\alpha \longmapsto [D] \cap \alpha$
(rel. cap product $H_{n+k}(D,\partial D) \times H^i(D, \partial D) \to H_{n+k-i}(D)$)

Lemma: Let $E \to B$, D as above, $i : B \hookrightarrow D$ incl of zero section.
Then $i_*[B] = [D] \cap u \in H_k(D, \mathbb{Z})$.

i.e. $i_*[B]$ = zero section is Poincaré dual to Thom class.

Pf: wlog assume B connected. Then we have isomorphisms

$$\mathbb{Z} = H^0(B, \mathbb{Z}) \xrightarrow[\text{Thom iso}]{\pi^*(\cdot) \cup u} H^n(D, \partial D; \mathbb{Z}) \xrightarrow[\text{Poincaré duality}]{[D]\cap} H_k(D, \mathbb{Z}) \underset{i_*}{\overset{\pi_*}{\rightleftarrows}} H_k(B, \mathbb{Z}) \simeq \mathbb{Z}$$

Since these are isoms, $1 \longmapsto \pm 1$ so $[D] \cap u = \pm\, i_*[B]$, check signs... $\blacktriangle$

Now, for $A \subset M$ submfld of codim. i, let N = tubular nbd of A ($\simeq$ normal bundle)
$\simeq$ rank i oriented vector bundle $/A$

$$\begin{tikzcd}
u_A \in H^i(N, \partial N; \mathbb{Z}) \arrow[d, "{[N]\cap -}"', "\text{Poincaré duality in } N"'] & H^i(M, M - \text{int } N; \mathbb{Z}) \arrow[l, "\simeq"', "\text{excision}"] \arrow[r] \arrow[d, "{[M]\cap -}"] & H^i(M, \mathbb{Z}) \arrow[dl, "\text{Poincaré duality in } M"] \\
{[A] \in H_{n-i}(N; \mathbb{Z})} \arrow[r, "\text{incl.}"'] & H_{n-i}(M, \mathbb{Z}) &
\end{tikzcd}$$

by lemma

<u>Commutes</u> by <u>naturality of cap product</u>

(using that fund. classes map to each other $(N, \partial N) \longrightarrow (M, M - \text{int } N) \longleftarrow M$
$[N] \longmapsto [M] \longleftarrow\!\!\shortmid [M]$)

→ **Prop:** ‖ $u_A^M :=$ image of Thom class of $N_{A/M}$ in $H^i(M,\mathbb{Z})$ is P.dual to $[A]\in H_{n-i}(M)$.
i.e. $[M]\cap u_A^M = [A]$

(where $A\subset M$ closed oriented submfld, $N_{A/M}$ oriented so fiber $\oplus$ base $\simeq$ orient of M).

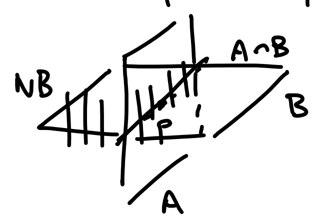
PF Thm: Recall: $A, B\subset M$ transversely intersecting, oriented; orient $A\cap B$

Orient normal bundles so that (fiber, base) orient^{n} matches with ambient space.

Observe: transversality $\Rightarrow$ $N_{A\cap B/A} \simeq$ restr. to $A\cap B$ to $N_{B/M}$

(since $T_pA \underset{\uparrow}{+} T_pB = T_pM \Rightarrow$ (complement to $T_p(A\cap B)$ in T_pA) $\oplus\, T_pB = T_pM$

intersect along $T_p(A\cap B)$



Hence, by naturality of Thom class,

$$u^A_{A\cap B} = i^*\left(u^M_B\right) \quad \text{under} \quad i^*: H^j(M,\mathbb{Z})\to H^j(A,\mathbb{Z})$$

(coming from $i^*: H^j(N_{B/M}, N^0_{B/M}) \to H^j(N_{A\cap B/A}, N^0_{A\cap B/A})$
under excision + map to absolute cohomology).

Prop for $A\cap B\subset M$

$$
\begin{aligned}
\text{Now: } [M]\cap u^M_{A\cap B} &= [A\cap B] \in H_{n-i-j}(M) \\
&= i_*\left([A]\cap u^A_{A\cap B}\right) \quad \text{where } i: A\hookrightarrow M
\end{aligned}
$$

Prop. for $A\cap B\subset A$

$$
\begin{aligned}
&= i_*\left([A]\cap i^*(u^M_B)\right) \\
&= (i_*[A])\cap u^M_B \\
&= \left([M]\cap u^M_A\right)\cap u^M_B \quad \longleftarrow \text{Prop for } A\subset M \\
&= [M]\cap\left(u^M_A \cup u^M_B\right) \quad \text{(elementary property of cap/cup)}
\end{aligned}
$$

i.e. $u^M_{A\cap B} = u^M_A \cup u^M_B$, which is the statement.

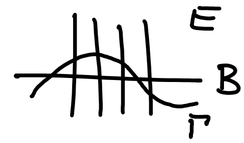
• Now, Poincaré duality interpretation of Euler class:

Recall $E\to B$ oriented rk n $\to$ $e(E)$ = image of Thom class under
$H^n(E, E-B,\mathbb{Z}) \to H^n(E,\mathbb{Z}) \underset{\pi^*}{\overset{\sim}{\leftarrow}} H^n(B,\mathbb{Z})$

Assume B smooth closed oriented mfld, and let $s : B \to E$ a smooth section.

$\Gamma = \{ s(x) \mid x \in B \} \subset E$ "graph of s"

$Z = s^{-1}(0) = \Gamma \cap \underset{\text{zero sec.}}{B}$ zero set of s



For generic choice of s, Γ is transverse to the zero section, so Z is a submanifold of B, of codim $=$ rank(E).

<u>Fact:</u> The derivative of s gives an isomorphism $N_{Z/B} \simeq E_{|Z}$

indeed: at $x \in Z$, $\quad T_xB \xrightarrow[D_x s]{} T_xE = T_xB \oplus E_x$ has image $T_x\Gamma$

$D_x s$ is of the form (id, ℓ_x) where the second factor is surjective by transversality.

$T_xZ = T_xB \cap T_x\Gamma = \ker(T_xB \overset{\ell_x}{\twoheadrightarrow} E_x)$, so $0 \to T_xZ \xrightarrow{incl} T_xB \to E_x \to 0$

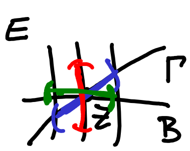
which gives $E_x \simeq T_xB / T_xZ = N_xZ$. $\blacktriangle$

Since E oriented, we use this isom. to orient $N_{Z/B}$, and then since B oriented this induces an orientation of Z.

<u>Thm:</u> $E \to B$, $Z = s^{-1}(0) \subset B$ as above. Then $e(E) = [Z]^* \in H^n(B, \mathbb{Z})$
Poincaré duality.

<u>Pf:</u> Let $u \in H^n(E, E-B; \mathbb{Z})$ Thom class of E, $u_{|Z} \in H^n(E_{|Z}, E^0_{|Z})$ restr. $=$ Thom class of $E_{|Z}$

Let $N =$ tubular nbd of Z in B, then $(N, N-Z) \simeq (N_{Z/B}, N^0_{Z/B}) \simeq (E_{|Z}, E^0_{|Z})$
$\hookrightarrow (E, E^0)$



is homotopic through maps of pairs to $(N, N-Z) \xrightarrow{s_{|N}} (E, E^0)$

so $(s_{|N})^*(u) = u_{|Z} \in H^n(N, N-Z; \mathbb{Z})$

By excision $H^n(N, N-Z; \mathbb{Z}) \simeq H^n(B, B-Z; \mathbb{Z})$ & can take image in $H^n(B, \mathbb{Z})$

Under this, $\underset{s_{|N} : (N, N-Z) \to (E, E^0)}{(s_{|N})^*(u)} \longmapsto \underset{s : (B, B-Z) \to (E, E^0)}{s^*(u)} \longmapsto \underset{s : B \to E \to (E, E_0)}{s^*(u)} \overset{\text{by def.}}{=:} e(E)$

isom in H^* $\quad H^*(E, E_0) \to H^*(E)$

but on the other side, $u_{|Z} \longmapsto u_Z^B \in H^n(B, \mathbb{Z})$
which by prop. is PD to $[Z]$. $\blacktriangle$